LINEAR PATTERN MATCHING ALGORITHMS

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Abstract

In 1970, Knuth, Pratt, and Morris [1] showed how to do basic pattern matching in linear time. Related problems, such as those discussed in [4], have previously been solved by efficient but sub-optimal algorithms. In this paper, we introduce an interesting data structure called a bi-tree. A linear time algorithm for obtaining a compacted version of a bi-tree associated with a given string is presented. With this construction as the basic tool, we indicate how to solve several pattern matching problems, including some from [4], in linear time.

I. Introduction

In 1970, Knuth, Morris, and Pratt [1-2] showed how to match a given pattern into another given string in time proportional to the sum of the lengths of the pattern and string. Their algorithm was derived from a result of Cook [3] that the 2-way deterministic pushdown languages are recognizable on a random access machine in time $O(n)$. Since 1970, attention has been given to several related problems in pattern matching [4-6], but the algorithms developed in these investigations usually run in time which is slightly worse than linear, for example $O(n \log n)$. It is of considerable interest to either establish that there exists a non-linear lower bound on the run time of all algorithms which solve a given pattern matching problem, or to exhibit an algorithm whose run time is of $O(n)$.

In the following sections, we introduce an interesting data structure, called a bi-tree, and show how an efficient calculation of a bi-tree can be applied to the linear-time (and linear-space) solution of several pattern matching problems.

II. Strings, Trees, and Bi-Trees

In this paper, both patterns and strings are finite length, fully specified sequences of symbols over a finite alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$. Such a pattern of length $m$ will be denoted as

$$ P = P(1)P(2) \ldots P(m), $$

where $P(i)$, an element of $T$, is the $i$th symbol in the sequence, and is said to be located in the $i$th position. To represent the substring of characters which begins at position $i$ of $P$ and ends at position $j$, we write $P(i:j)$. That is, when $i \leq j$, $P(i:j) = P(i) \ldots P(j)$, and $P(i:j) = \lambda$, the null string, for $i > j$.

Let $\Sigma^*$ denote the set of all finite length strings over $\Sigma$. Two strings $\omega_1$ and $\omega_2$ in $\Sigma^*$ may be combined by the operation of concatenation to form a new string $\omega = \omega_1 \omega_2$. The reverse of a string $P = A(1) \ldots A(m)$ is the string $P^* = A(m) \ldots A(1)$. The length of a string or pattern, denoted by $|P|$ for $P \in \Sigma^*$, is the number of symbols in the sequence. For example, $|P(i:j)| = j-i+1$ if $1 \leq i \leq j$ and is 0 if $i > j$.

Informally, a bi-tree over $\Sigma$ can be thought of as two related t-ary trees sharing a common node set.

Before giving a formal definition of a bi-tree, we review basic definitions and terminology concerning t-ary trees. (See Knuth [7] for further details.)

A t-ary tree $T$ over $\Sigma = \{\sigma_1, \ldots, \sigma_t\}$ is a set of nodes $N$ which is either empty or consists of a root, $n_0 \in N$, and $t$ ordered, disjoint t-ary trees. Clearly, every node $n_i \in N$ is the root of some t-ary tree $T_i$, which itself consists of $n_i$ and $t$ ordered, disjoint t-ary trees, say $T_i^1, T_i^2, \ldots, T_i^t$. We call the tree $T_i$ a sub-tree of $T_i^j$; also, all sub-trees of $T_i$ are considered to be sub-trees of $T_i^j$.

It is natural to associate with a tree $T$ a successor function

$$ S : \Sigma^* \rightarrow (N - \{n_0\}) \cup \{\text{NIL}\} $$

defined for all $n_i \in N$ and $\sigma_j \in \Sigma$ by

$$ S(n_i, \sigma_j) = \begin{cases} n_i' & \text{if } T_i^j \text{ is non-empty} \\ \text{NIL} & \text{if } T_i^j \text{ is empty} \end{cases} $$

It is easily seen that this function completely determines a t-ary tree and we write $T = (N, n_0, S)$.

If $n' = S(n, \sigma)$, we say that $n$ and $n'$ are connected by a branch from $n$ to $n'$ which has a label of $\sigma$. We call $n'$ a son of $n$, and $n$ the father of $n'$. The degree of a node $n$ is the number of sons of that node, that is, the number of distinct $\sigma$ for which $S(n, \sigma) \neq \text{NIL}$. A node of degree 0 is a leaf of the tree.

It is useful to extend the domain of $S$ from $\Sigma^*$ to $(N \cup \{\text{NIL}\}) \times \Sigma^*$ (and extend the range to include $n_0$) by the inductive definition

- (S1) $S(\text{NIL}, \omega) = \text{NIL}$ for all $\omega \in \Sigma^*$
- (S2) $S(n, \lambda) = n$ for all $n \in N$
- (S3) $S(n, \omega \sigma) = S(S(n, \omega), \sigma)$ for all $n \in N$, $\omega \in \Sigma^*$, and $\sigma \in \Sigma$.

Not every $S : \Sigma^* \rightarrow (N - \{n_0\}) \cup \{\text{NIL}\}$ is the successor function of a t-ary tree. But a necessary and sufficient condition for $S$ to be a successor function of some (unique, if it exists) t-ary tree can be expressed in terms of the extended $S$. Namely, that there exists exactly one choice of $\omega$ such that $S(n_0, \omega) = n$ for every $n \in N$. When there exists a $T$ such that $T = (N, n_0, S)$, we say that $T$ is legitimate.

We may also associate with $\tau$ a father function $F : N \rightarrow N$ defined by $F(n_0) = n_0$ and for $n' \in N - \{n_0\}$,

$$ F(n') = n \iff S(n, \sigma) = n' \text{ for some } \sigma \in \Sigma. $$

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Let \( F^O(n) \equiv n \) for all \( n \in N \). It may be shown that the \( k \)-fold composition of \( F, F^k \), for positive \( k \) and \( n \neq n_0 \), satisfies \( F^k(n) \neq n \), and that for any \( n \) there exists a least value of \( k \) such that \( F^k(n) = n_0 \). This value is called the level of the node. Any \( n' = F^k(n) \) for positive \( k \) is said to be an ancestor of \( n \). (The root \( n_0 \) is an ancestor of all other nodes in the tree.)

There is another important function which may be associated with a \( t \)-ary tree \( T \) over the alphabet \( \Sigma \). This function \( W : N \rightarrow \Sigma^* \) associates a string of symbols from \( \Sigma \) with each node of \( T \), and is defined recursively by

\[
(W1) \quad W(n_0) = \Lambda
\]

\[
(W2) \quad W(n) = W(n')\cdot\sigma \Leftrightarrow n = S(n',\sigma).
\]

It is not hard to show that \( W1 \) and \( W2 \) completely specify a well-defined function, and moreover that the sequence of branches which connect the root to any other node \( n \) in \( T \) are labelled with the elements of \( W(n) \). (The label of the branch from \( n_0 \) is the leftmost element of \( W(n) \), etc.) It is also possible to show that the length of \( W(n) \) equals the level of node \( n \). Indeed, an inductive argument can be made to establish the useful assertion that for all \( n \in N \) and \( \omega \in \Sigma^* \),

\[
\omega = W(n) \Rightarrow n = S(n_0,\omega)
\]
as well as the identity

\[
n_0 = F^L(p(W(n)))(n) \text{ for all } n \in N.
\]

Note also that when \( S \) is not legitimate, the function \( W \) defined recursively in terms of \( S \) by \( W1 \) and \( W2 \) is not well defined. Thus, \((N, n_0, S)\) is a \( t \)-ary tree if and only if \( W \) is well defined. We call the function \( W \) the \emph{walk function} associated with \( T \).

The association of a node \( n \) with the string \( \omega = W(n) \) is an important one. In order to be able to associate \( \omega \) with \( n \) directly we adopt the following notational convention. If \( n' \) is a node in \( N \), we write \( \omega' = W(n') \). Similarly, write \( \omega = W(\sigma) \) for \( n \in N \), \( \omega = W(\sigma') \) for \( n' \in N \), etc.

**Definition:**

A bi-tree \( B = (N, n_0, S_p, S_s) \) over the alphabet \( \Sigma = \{\sigma_1, \ldots, \sigma_t\} \) is a set of nodes \( N \) with a designated root \( n_0 \in N \), together with the functions

\[
S_p : N \times \Sigma \rightarrow \{N - \{n_0\}\} \cup \{NIL\}
\]

and

\[
S_s : N \times \Sigma \rightarrow \{N - \{n_0\}\} \cup \{NIL\}
\]
such that

\[
(B1) \quad T_p = (N, n_0, S_p) \text{ is a } t \text{-ary tree,}
\]

\[
(B2) \quad T_s = (N, n_0, S_s) \text{ is a } t \text{-ary tree, and}
\]

\[
(B3) \quad W_p(n) = [W_s(n)]^T \text{ for all } n \in N,
\]

where \( W_p(n) \) is the walk function associated with the tree \( T_p \), and \( [W_s(n)]^T \) is the reverse of \( W_s(n) \).

We call the tree \( T_p \) the \( p \)-tree associated with \( B \), and the tree \( T_s \) the \( s \)-tree associated with \( B \). We also say that the bi-tree \( B \) is an \( s \)-extension (\( p \)-extension) of its \( p \)-tree (\( s \)-tree). When appropriate to prevent confusion, we use terms such as \( p \)-branch to indicate a branch of the \( p \)-tree, etc.; also, the function \( F^L(p) \) is the father function of the tree \( T_p \). However, if a term or function is written without an \( s \) or \( p \) identifier, we mean to refer to the \( p \)-tree concept.

**Remark:**

It follows from the definition of a bi-tree that if a node is at the \( j \)th level of the \( p \)-tree it must also be at the \( j \)th level of the \( s \)-tree, and vice-versa. Actually, the \( p \)-tree and \( s \)-tree are anti-isomorphic images of one another in the sense of [4].

The definition of a bi-tree does not in itself insure that there exist any bi-trees at all; however, an example of a bi-tree is shown in Figure 1, which establishes that the definition is non-vacuous.

A useful relationship between the extended functions \( S_p \) and \( S_s \) of a bi-tree is provided in the following lemma.

**Lemma 1:**

Let \( B = (N, n_0, S_p, S_s) \) be a bi-tree over \( \Sigma \). Then for all \( n \in N \) and \( \omega \in \Sigma^* \),

\[
n = S_p(n_0,\omega) \Leftrightarrow n = S_s(n_0,\omega^T).
\]

**Proof:**

Consider the string \( \omega = W_p(n) \). Since \((N, n_0, S_p)\) is a \( t \)-ary tree,

\[
\omega = W_p(n) \Leftrightarrow n = S_p(n_0,\omega).
\]

Similarly, since \((N, n_0, S_s)\) is a \( t \)-ary tree,

\[
\omega = W_s(n) \Leftrightarrow n = S_s(n_0,\omega).
\]

We also have, from the definition of a bi-tree, that

\[
W_p(n) = [W_s(n)]^T.
\]

It follows that

\[
n = S_p(n_0,\omega) \Leftrightarrow \omega = W_p(n)
\]

\[
\omega^T = W_s(n)
\]

\[
\Rightarrow n = S_s(n_0,\omega^T).
\]

QED

If \( T \) is a given \( t \)-ary tree, there may or may not exist a bi-tree which is either an \( s \)-extension or \( p \)-extension of \( T \). (Of course, the symmetry of the definition implies that if \( T \) has an \( s \)-extension \( B \) then it also has a \( p \)-extension \( B' \), and vice-versa.)

**Theorem 1:**

A given \( t \)-ary tree \( T = (N, n_0, S) \) is the \( p \)-tree associated with some bi-tree \( B \) if and only if all \( n \in N \), \( \sigma \in \Sigma \), and \( \omega \in \Sigma^* \),

\[
n = S(n_0,\omega) \Rightarrow \text{there exists a } n' \in N \text{ such that } n' = S(n_0,\omega)\) (2).
\]

**Proof:**

Suppose that \( T \) is the \( p \)-tree associated with the
Lemma 1 also implies that \( n' = S_p(n_0, \omega) \), and (2) is established.

Now assume that (2) holds. Let \( W \) denote the walk function of \( T \), and define \( B = (N, n_0, S_p, S_s) \) by \( S_p = S \) and for all \( n \in N \) and \( \sigma \in \Sigma \),

\[
S_p(n, \sigma) = S_p(n_0, \sigma)(n_\sigma).
\]

We must establish that (B1), (B2), and (B3) hold for \( B \). Certainly, \( T_p = (N, n_0, S_p) \) is a t-ary tree. To show that \( T_s = (N, n_0, S_s) \) is a t-ary tree, and to establish (B3), we prove by induction that the function \( W_s \) defined inductively in terms of \( S_s \) by (W1) and (W2) above is well defined and satisfies

\[
(W_3) \quad [W_s(n)]^r = W_p(n) \text{ for all } n \in N.
\]

The induction is on the length of \( W_p(n) \). If \( \ell_q(W_p(n)) = 0 \), then \( n = n_0 \), and from (W1) we have \( [W_s(n_0)]^r = \Lambda = W_p(n_0) \). Also, since the range of \( S_s \) does not include \( n_0 \), the value of \( W_s(n_0) \) is well defined by (W1) and (W2).

The inductive hypothesis is that if \( n' = \text{any node with } \ell_q(W_p(n')) = k \), then (B3) holds for this node, and that the value of \( W_s \) obtained from (3), (W1) and (W2) is well defined. Let \( n \) be a node such that \( W_p(n) = \omega \), where \( \ell_q(W_p(n)) = k \). From (1) we have that \( n = S_p(n_0, \omega) \), so (2) implies that there exists a (unique) \( n' \in N \) such that \( n' = S_p(n_0, \omega) \). Using (1) once more, we obtain \( \omega = W_p(n') \). Thus, from (3), \( S_p(n', \omega) = S_p(n_0, \omega) = n \). The inductive hypothesis allows that \( W_s(n') \) is well defined, and since (W2) defines \( W_s(n) \) in terms of \( W_s(n') \), we can see that \( W_s(n) \) is also well defined. The inductive hypothesis also establishes that \( W_s(n') = [W_s(n')]^r = \omega^r \). Finally, we may deduce that \( W_s(n) = W_s(n') = \omega^r = [W_p(n)]^r \). This completes the induction and the proof.

QED

We now relate the concept of a bi-tree to that of a string. First, however, consider the basic problem of finding a match of a given pattern \( P \) of length \( k \) with another string \( S \) of length \( k' \), where \( k' \geq k \). That is, find positions \( i \) and \( j \) within \( S \) such that \( P = \sigma \). Clearly, if \( i \) or \( P \) does not match some substring of \( S \), then \( P = P(k+1) ... P(L) \), the reverse of \( P \), also matches a substring of \( S \). This observation implies that every technique which solves a pattern matching problem working from left to right has a dual procedure which works from right to left. In what follows, we adopt a left to right viewpoint, referring only briefly to dual concepts as appropriate.

With this understanding, we henceforth assume (for purely technical reasons) that every string \( S \in \Sigma^* \) ends in a symbol which does not occur elsewhere in \( S \). Also, when we refer to the substring located at position \( i \) of \( S \), we mean that \( S(i) \) is the leftmost symbol of the substring.

Definition:

The prefix-tree associated with string \( S \) over \( \Sigma = \{\sigma_1, ... , \sigma_t\} \) is a t-ary tree \( T_p = (N, n_0, S_p) \) with exactly \( \ell_q(S) \) leaves such that there is a bijective pointer function \( J \) from the set of leaves of \( T_p \) to the set of positions within \( S \) such that if \( j = J(n) \) then \( W_p(n) \) is the minimal length unique substring of \( S \) whose leftmost symbol is located at position \( j \) of \( S \). That is, \( W_p(n) = \sigma(j:k) \) occurs only once in \( S \) and \( \sigma(j:k-1) \) occurs at least twice in \( S \). We call \( I(n) = \sigma(j:k) \) the prefix identifier associated with position \( j \).

(The concept of suffix tree may be similarly defined for strings with left endmarkers.)

The assumption that \( S \) has a unique endmarker on the right insures that every position of \( S \) has a prefix identifier. This implies that there is exactly one prefix-tree associated with a given \( S \). Moreover, if \( n \) is any node of the prefix-tree of \( S \), then \( W_p(n) \) is a substring of \( S \). Indeed, the substring \( W_p(n) \) occurs at every position \( J(n') \) of \( S \) such that \( n' \) is a leaf of the subtree whose root is \( n \). Consequently, the minimality condition concerning \( W_p(n) \) implies that the index of any father of a leaf is greater than one, and that every leaf node has a brother.

Figure 2 shows the prefix-tree associated with the string \( S = 0110101 \).

As may be surmised from our choice of terminology, the prefix-tree associated with any string \( S \) has an s-extension.

Theorem 2:

For every string \( S \) over \( \Sigma \), there exists a bi-tree of \( S \), \( B_p = (N, n_0, S_p, S_s) \) such that \( (N, n_0, S_p) \) is the prefix-tree associated with \( S \).

Proof:

Consider the prefix tree \( T_p = (N, n_0, S_p) \) associated with \( S \). We first show that if node \( n \in N \) is equal to \( S(n_0, \omega) \) for some \( \sigma \in \Sigma \), \( \omega \in \Sigma^* \), then there exists a node \( n' = S_p(n_0, \omega) \in N \). The assumption that \( n = S_p(n_0, \omega) \) implies that \( W_p(n) = \omega \), so \( \omega \) must be a substring of \( S \). Moreover, either \( \sigma \) occurs at least twice in \( S \), or \( \sigma \) is a prefix identifier of \( S \). If \( \sigma \) occurs more than once, then \( \omega \) must be a substring of \( S \) and \( \sigma \) is a prefix identifier or \( \omega \) occurs more than once, since every prefix of \( \sigma \) must occur more than once. In either case, there is a node \( n' \) with \( W_p(n') = \omega \) and \( S_p(n_0, \omega) = n' \). Theorem 1 can now be directly invoked to complete the proof.

QED

From the proof of Theorem 1, recall that the s-tree \( T_s = (N, n_0, s_s) \) of \( B_p \) is defined by (3). We call the bi-tree \( B_p \) the prefix bi-tree associated with \( S \). (It is also true that there exists a suffix bi-tree associated with every string \( S \) with a left endmarker.)

As will be shown in Section IV, linear-time and linear-space algorithms for certain pattern matching problems can be derived assuming that an appropriate prefix-tree is pre-calculated. We have been unable to find efficient methods for directly obtaining a prefix-tree. But as we show in the next section, efficient methods exist for calculating the prefix bi-tree whose p-tree is the desired prefix-tree; more important, a linear-time, linear-space algorithm for obtaining a compacted prefix bi-tree will be exhibited.
III. Computation of Prefix Bi-Trees and Compacted Prefix Bi-Trees

It is well to consider first a direct method for obtaining the prefix-tree associated with a given string S of length m.

Our direct method is an iteration of an algorithm to compute the prefix-tree T_i of the suffix substring S_i = S(i:m) assuming that the prefix-tree T_{i+1} of the suffix substring S_{i+1} = S(i+1:m) is known. The following lemma provides the theory which both motivates the algorithm and which can be used to prove its correctness. Its usefulness in this regard is based on the observation that the prefix-tree T of a string S is completely determined by the set I = {I(j)|1 ≤ j ≤ m} of prefix identifiers.

**Lemma 2:**

Let I^{i+1} = {I_{i+1}^{j+1})|1 < j ≤ m} be the set of prefix identifiers of S_{i+1} = S(i+1:m) and let I_i be the set of prefix identifiers of S_i. Define ñ to be the longest prefix of S_i which is also a prefix of some element of I^{i+1}. If ñ ≠ I_{i+1}(j) for some j, then I_i(j) = I_{i+1}(j) for i < j ≤ m and I_i(j) = ñ S(t+lg(ñ)). Otherwise, if ñ = I_{i+1}(j), then I_i(j) = I_{i+1}(j) when i < j ≤ m and j ≠ j_0, I_{i+1}(j_0) = ñ = S(j_0, j_0+lg(ñ)-1) and I_i(j) = S(t+1lg(ñ)-1), where ñ is the shortest prefix of S_j_0 which does not equal a prefix of S_i.

**Proof:**

Suppose first that ñ is not an element of I_{i+1}, and consider the prefix of S_i given by ñ S(t+lg(ñ)). This string is the shortest prefix of S_i which is not a prefix of any element of I_{i+1}; conversely, no element of I_{i+1} is a prefix of this string. Thus the prefix identifiers of S_i are given by I_i(j) = I_{i+1}(j) for 1 < j ≤ m and I_i(j) = ñ S(t+lg(ñ)).

Now suppose that ñ = I_{i+1}(j_0). Surely, ñ is not a prefix of any other member of I_{i+1}, nor can any member of I_{i+1} other than I_{i+1}(j_0) be a prefix of ñ. This ensures that for i < j ≤ m and j ≠ j_0, I_i(j) = I_{i+1}(j). I_{i+1}(j_0) and I_i(j) must each be of length one greater than the length of their largest common prefix, say ñ. Thus, I_i(j_0) = ñ S(t+lg(ñ)) = S(j_0, j_0+lg(ñ)-1) and I_i(j) = ñ S(t+lg(ñ)) = S(t+1lg(ñ)-1). Clearly, ñ is the shortest prefix of S_j_0 which does not equal a prefix of S_i.

**QED**

The two cases described in the proof of Lemma 2 have interesting analogies in the tree T_i+1. In the first case ñ is an internal node and T_i may be formed by adding the leaf n_j (with J(n_j) = j) to T_{i+1} by connecting it to ñ with a branch labelled S(t+lg(ñ)). In the second case, ñ is the leaf with J(ñ) = j_0. Here, ñ must be replaced by a two-leaf subtree rooted at ñ. The node ñ is the leaf with J(ñ) = j_0. The other leaf of the subtree, n_{j+1}, has J(n_{j+1}) = 1. These two nodes, as well as any other nodes in the subtree between the root ñ and the two leaves must also be added to T_{i+1} to form T_{i+1}. In the first case, we say that T_i is obtained from T_{i+1} by a type 1 construction; in the second case, by a type 2 construction. It is also useful to distinguish three subcases of a type 2 construction: 1a) ñ is the father of ñ, 1b) ñ is the father of ñ (the father of ñ), and 2c) ñ is an ancestor (but not the father) of ñ.

Figure 3 illustrates these cases and our notational conventions.

In all cases, the calculation of T_i from T_{i+1} suggested by the Lemma first locates the node ñ. Algorithm D, below, implements this calculation by walking from the root 0 by traversing the branches which are labelled with symbols from the prefix of S_i.

We assume that at the beginning of the algorithm T_1 is available as a set of nodes N_1 and a function S defined for N_1 × Σ. The construction of T_i forms N_i by adding nodes to N_{i+1}. (When a new node is added, we assume that all unspecified values of S, etc., are initialized to NIL.) It is also convenient to associate with each node n a label J(n) which is the position in S_{i+1} of the (rightmost, if not unique) substring ω = W(n). (Note that J(n) is consistent with the pointer function J(n) in that if n is a leaf, J(n) = J(n)).

Algorithm D (Direct construction of T_i from T_{i+1})

D1. [Initialize] Set ñ = n_k and j = 0.

D2. [Find ñ] If S(ñ, S(j+k)) = NIL go to D3; otherwise, set ñ + S(ñ, S(j+k)), increment k, and repeat step D2. (When D3 is entered, ñ = W(n).)

D3. [Is ñ a leaf?] Set ñ + ñ. If S(ñ, S(j,n+k)) NIL go to D6. (This is the case that ñ is not a prefix identifier of T_{i+1}.)

D4. [Move leaf] Add node ñ and connect to ñ by setting S(ñ, S(j,n+k)) + n. Label the leaf by setting J(n) + J(n).

D5. [Compare symbols of S] If S(j+k) equals S(j,n+k) set ñ + ñ, increment k, and go to D4.

D6. [Add n_k to tree] Add node n_k and connect to ñ by setting S(ñ, S(j,n+k)) + n_k. Label the leaf by setting J(n_k) + J(n_k).

The prefix-tree of S can be obtained by successively deriving the prefix-trees of S_m, S_{m-1}, ..., S_2, S_1 = S. Figure 4 shows several steps in a typical calculation.

Note that at each iteration as many as O(lg(ñ)) steps may be required. There are m iterations, and it may easily be shown that the total number of steps can be of O(m^2). What's worse, it may be seen that this algorithm is at best O(m log m).

We now turn to the problem of finding the prefix bi-tree B^P_1 corresponding to S_1 given the prefix bi-tree B^P_{i+1}. Our method is based on a relationship between T_{i+1}(i+1) and I_i(i) which is described in the following lemma.
Lemma 3:

Suppose that the symbol \( S(\ell) \) occurs within the string \( S_{i+1} \) and let \( \mathcal{W}' \) be the longest prefix of \( I_{i+1}(i+1) \) which occurs at some position other than \( i+1 \), say \( j_0+1 \), such that \( S(j_0) = S(\ell). \) Then \( I_1(1) = S(\ell)W(\mathcal{W}'(i+lg(W'))+1). \)

Proof:

Since \( I_{i+1}(i+1) \) is a prefix identifier of \( S_{i+1} \), it can occur only once in \( S_{i+1} \) and clearly \( S(\ell)I_{i+1}(i+1) \) also occurs exactly once in \( S_1 \). Thus \( I_1(1) \) is of the form \( S(\ell)w_1^i \), where \( w_1^i \) is a prefix of \( I_{i+1}(i+1) \). Since we have assumed that \( S(\ell) \) occurs within \( S_{i+1} \), \( I_1(1) = S(\ell)w_1^i \), \( \ell = 1 \), and we can write \( \mathcal{W}' = S(\ell)W(\mathcal{W}'(i+lg(W'))+1) \) and be sure that \( \mathcal{W}' \) occurs at least twice in \( I_{i+1}(i+1) \). This implies that \( \mathcal{W}' \) is the longest prefix of \( I_{i+1}(i+1) \) which occurs at some position other than \( i+1 \), say \( j_0+1 \), such that \( S(j_0) = S(\ell) \). We have just established that \( I_1(1) = S(\ell)w_1^i = S(\ell)S(\ell)W(\mathcal{W}'(i+lg(W'))+1). \)

QED

In those situations where \( S(\ell) \) does not occur within \( S_{i+1} \), it is trivial to show that \( I_1(1) = S(\ell) \).

We now wish to draw out some important relationships between the strings defined in the preceding lemmas and proofs.

In the proof of Lemma 2, we have used \( \mathcal{H} \) to represent the father node of \( \mathcal{N}_d \), where \( \omega_d = I_{i+1}(i) = W_p(n) \) in \( B_{i+1}^p \). It follows from Lemma 3 that \( \mathcal{W} = S(\ell)\mathcal{W}' \), where \( \mathcal{W}' \) is the longest prefix of \( \omega_d = I_{i+1}(i+1) \) which occurs elsewhere in \( S_{i+1} \) with \( S(\ell) \) to its left. This relationship implies that \( \mathcal{H} \) and \( \mathcal{N}_d \) are related by \( \mathcal{H} = S_{\mathcal{W}}(\mathcal{W}',S(\ell)) \) in the prefix bi-tree \( B_{i+1}^p \). Similarly, if \( \mathcal{N}_d = S(\hat{\mathcal{W}},\sigma) \), where \( \sigma = S(\hat{\mathcal{W}}(i+lg(\hat{\mathcal{W}}')+1)) \), it may be seen that \( \hat{\mathcal{W}} = S_{\mathcal{W}}(\mathcal{W}',S(\ell)) \). Indeed, all ancestors of \( \mathcal{N}_d \) except for \( n_0 \) are s-sons of ancestors of \( \mathcal{N}_d \) with branches labelled by \( S(\ell) \). In particular, the node \( \mathcal{N}_d \) which is the closest ancestor of \( \mathcal{N}_d \) having a non-NIL s-son labelled \( \mathcal{N}_d \) in \( B_{i+1}^p \) plays an important role in forming \( B_{i+1}^p \) since all ancestors of \( \mathcal{N}_d \) which are descendants of \( \mathcal{N}_d \) will be s-fathers of nodes in \( B_{i+1}^p \) that do not exist in \( B_{i+1}^p \). Note that the situation \( \mathcal{N}_d \) corresponds to the condition \( \omega_d = I_{i+1}(i) \), and \( n_d \) is the only node in \( B_{i+1}^p \) which is not in \( B_{i+1}^p \). (This requires a type 1 construction.) Also, when \( \mathcal{N}_d \) is an ancestor of \( \mathcal{N}_d \), a sub-tree with leaves \( n_d \) of \( \mathcal{N}_d \) must be added to \( B_{i+1}^p \) to obtain \( B_{i+1}^p \). (This requires a type 2 construction.) This sub-tree is rooted at node \( \mathcal{N}_d \) and \( \mathcal{N}_d(8) = \mathcal{N}_d(8)(i) \). In all cases, \( n_d \) is \( \mathcal{N}_d \), where \( n_d = \mathcal{N}_d(8), \sigma \) for \( \sigma = S(\hat{\mathcal{W}}(i+lg(\hat{\mathcal{W}}')+1)) \). Figure 5 illustrates the notation by showing part of the prefix bi-trees \( B_{i+1}^p \) and \( B_{i+1}^p \).

By convention, we assume that the empty string, \( \Lambda \), occurs at every position of \( S_{i+1} \).

Lemma 3 and the relationships just described suggest that each node of the prefix bi-tree \( B_{i+1}^p \) be labelled with a t-long vector \( L_n^p \). The \( j \)-th component \( L_n^p(j) \), \( 1 \leq j \leq t \), is set equal to NIL if the substring \( S_j(n) \) does not occur within \( S_{i+1} \); if \( S_j(n) \) does occur within \( S_{i+1} \), then \( L_n^p(j) \) is the (rightmost, if not unique) position of \( S_{i+1} \) such that \( S_j(n) \) is a prefix of \( S_{i+1} \).

Algorithm B, below, calculates a labelled \( B_{i+1}^p \) from a labelled \( B_{i+1}^p \). It begins by walking from node \( n_{i+1} \) towards the root until the node \( n_{i+1} \) is reached. This node is identified by the fact that it is the first node reached with a non-NIL \( L_n^p(j) \), where \( S(j) = S_j(n) \). The node below \( n_{i+1} \) on the path from \( n_{i+1} \) is the node \( n_0 \) and \( n_0 \) will eventually be installed as the s-son under \( n_0 \) and as a p-son of \( n_0 \). Also, if node \( \tilde{n} \) is not already present (as determined by examining the s-branch of \( \tilde{n} \) labelled \( S(\ell) \)) a sub-tree rooted at \( \tilde{n} \) is added to \( B_{i+1}^p \) to obtain \( B_{i+1}^p \). The node \( \tilde{n} \), which is the nearest ancestor of \( n_{i+1} \) already in \( B_{i+1}^p \), is located by walking from \( n_{i+1} \) towards the root until a node \( \tilde{n} \) is reached with a non-NIL s-branch labelled \( S(\ell) \). The node \( \tilde{n} \) is the s-son of \( n_{i+1} \) just found. It should now be clear that the primary role of the s-tree is to determine efficiently whether a node required in \( B_{i+1}^p \) is already present in \( B_{i+1}^p \).

All of the functions and labels used in Algorithm D are also used in Algorithm B. In addition, we use the vector label \( L_n^p \) as discussed, as well as the p-father function \( F \), the s-successor function \( S_{\mathcal{W}} \), and a label \( L_n^p \) which gives the value of \( I_1(1) \).

Algorithm B (Efficient construction of \( B_{i+1}^p \) from \( B_{i+1}^p \))

B1. [Initialize] Set \( \mathcal{N} = n_{i+1} \) and set \( j \) to the index of \( S(\ell) \). (\( n_{i+1} = I_{i+1}(i+1) \) and \( S(\ell) = S_j(n) \).)

B2. [Check label] If \( L_n^p(j) = \text{NIL} \) go to step B3; otherwise go to step B4. (When \( B_4 \) is entered, \( \mathcal{W} = W(\hat{\mathcal{W}}) \).

B3. [Label and walk] Set \( L_n^p(j) = I_{i+1}(i+1) \) and \( S_j(n) \) to go to step B2; otherwise set \( \tilde{n} = n_0 \) and go to step B7.

B4. [Find \( \tilde{n} \)] Set \( \tilde{n} = S(\hat{\mathcal{W}},S_{\mathcal{W}}(\mathcal{N},(i+1)+LG(\mathcal{N})) \).

B5. [Is \( \tilde{n} \) in \( B_{i+1}^p \)?] If \( S_{\mathcal{W}}(\tilde{n},S_j(n)) \neq \text{NIL} \), set \( \tilde{n} = S_{\mathcal{W}}(\tilde{n},S_j(n)) \) and go to step B7. (This is the case when \( \tilde{n} \) is not a prefix identifier of \( S_{i+1} \).

B6. [Add \( \tilde{n} \) and ancestors] If \( S_{\mathcal{W}}(\tilde{n},S_j(n)) \neq \text{NIL} \), use an implicit pushdown stack to save \( \tilde{n} \) and repeat this step recursively with \( \tilde{n} \) equal to \( F(\tilde{n}) \). (When this point is reached, \( S_{\mathcal{W}}(\tilde{n},S_j(n)) \neq \text{NIL} \).

Set \( \tilde{n} = S_{\mathcal{W}}(\mathcal{N},(i+1)+LG(\mathcal{N})) \) add node \( \tilde{n} \) by setting
As with our direct method, the prefix bi-tree of 

$$S(n, S(l+LG_n)) + \hat{n}, F(\hat{n}) + \hat{n},$$ and 

$$S_s(n', S_s) + \hat{n}.$$

Label \( \hat{n} \) by setting \( J_{\hat{n}} = J_{\hat{n}}, LG_{\hat{n}} = LG_{\hat{n}} + 1 \), and 

\( L_n(k) + L_n(k) \) for \( 1 \leq k \leq t \).

B7. [Add \( n_1 \)] Add node \( n_1 \) by setting 

\[ S(n, S(1+LG_n)) + n_1, F(n_1) + n_1, \text{ and } S_s(n_1, S_s) + n_1. \]

Label \( n_1 \) by setting 

\[ J_{n_1} = J_{n_1}, LG_{n_1} = LG_{n_1} + 1, \text{ and } L_{n_1}(k) + L_{n_1}(k)\text{ for } 1 \leq k \leq t. \]

Stop.

In order to derive a useful characterization of those non-NIL s-branches of \( B^p \) which also occur in \( C^p \), we present the following lemma.

**Lemma 4:**

Let \( n \neq n_0 \) be a node of \( B^p = (N_n, \eta_0', S_s) \) with degree \( d_n \) (in \( T_p \)) and let \( n' = F_s(n) \) have degree \( d_{n'} \).

Then \( d_{n'} \geq d_n \).

**Proof:**

If \( n \) is a leaf of \( T_p \), \( d_n = 0 \) and surely \( d_{n'} \geq d_n \).

Consider, then, a son of \( n \), say \( \hat{n} = S_s(n_{s}, \sigma) \), and let \( \hat{n}' = F_s(\hat{n}) \) with \( \hat{n} = S_s(n_{s}^{\prime}, \sigma) \).

From the definition of a bi-tree, it follows that \( \omega = \omega'(n) = \sigma'' \omega \), where \( \omega = \omega(n') \). But \( \omega(\hat{n}) = \sigma'' \omega', \) so \( \omega(\hat{n}') = \omega'(\sigma') \). It follows that \( S_s(n_{s}, \sigma') = \hat{n}' \). Thus, for every son of \( n \) there exists a son of \( n' \), and the lemma is established.

QED

Our desired characterization of \( S_s^C \) can now be established.

**Theorem 3:**

Let \( C^p = (N_n, \eta_0', S_s, S_s^C) \) be the compacted prefix bi-tree of the string \( S \). If \( n \neq n_0 \) is a node of \( C^p \), then there exists a node \( n' \in N_c^C \) and a \( \sigma \in \Sigma \) such that 

\[ S_s^C(n', \sigma) = n. \]

**Proof:**

Let \( B^p = (N_n, \eta_0, S_p, S_s) \) be the prefix bi-tree of \( S \), and let \( n' = F_s(n) \). If \( d_n \geq 2 \), then \( d_{n'} \geq 2 \). Let \( \hat{n} = F_s(n) \) and \( \hat{n}' = F_s(\hat{n}) = F_s(n) = F_s(n') \). If \( d_n \geq 2 \) then \( d_{n'} \geq 2 \), but \( n \) is a node in \( N_c^C \) and either \( d_n \geq 2 \) or \( d_{n'} \geq 2 \). It follows that node \( n' = F_s(n) \) is also a node in \( N_c^C \).

Clearly, there also exists a \( \sigma \in \Sigma \) such that 

\[ S_s^C(n', \sigma) = n. \]

QED

What we have just shown is that every \( n \neq n_0 \in N_c^C \) has both a p-father and an s-father. A consequence of this fact is that \( T^C = (N_c^C, \eta_0', S_s^C) \) is also a t-ary tree when both NIL and * values of \( S_s^C \) are taken to be pointers to empty sub-trees.

The difference between NIL values of \( S_s^C \) and * values of \( S_s^C \) is important to our algorithm for calculating \( C^p \). This procedure differs from Algorithm B in two important respects. First, when a type 2c construction is required, the father of \( n_1 \) and \( n, \hat{n} \), is connected
directly to \( \hat{n} \). Second, when a type 1 construction is required, but either \( \hat{n} \) or \( \hat{n} \) is not already present, an insertion is made between two nodes in the tree.

As before, let \( C^0 \) be the compacted prefix bi-tree of \( S_1 \). Algorithm C computes \( C^0 \) from \( C^0 \). (Note that \( L_g(n) \) gives the length of the walk \( L_g(W(n)) \) in the non-compacted prefix bi-tree.)

Algorithm C (Construction of \( C^0 \) from \( C^0 \))

C1. **Initialize** Set \( \hat{n}+1 + n+1 \) and set \( j \) to the index of \( S(i) \). \( (W(n+1) = I+1+I) \) and \( S(i) = S_j(i) \).

C2. **Check label** If \( L_g(j) = \text{NIL} \) go to step C3; otherwise go to step C4. (When C4 is entered, \( \hat{n} = W(\hat{n}') \).)

C3. **Label and walk** Set \( L_g(j) = j+1 \) and \( n+1 = n_+1 \). If \( n+1 \neq n_+1 \), set \( \hat{n} = F^2(n_+1) \) and go to step C2. Otherwise, set \( \hat{n} = n_+1 \) and go to step C14.

C4. **Find \( \hat{n} \)** Set \( \hat{n} = S^C(\hat{n}',S(\hat{n}'+1)+L_g(n)) \).

C5. **Is \( \hat{n} \) in \( C^0 \)?** If \( S^C(\hat{n}',S_j) \neq \text{NIL} \), set \( \hat{n} = S^C(\hat{n}',S_j) \) and go to step C8; otherwise, set \( \hat{n} = \hat{n} ' = F^2(\hat{n}') \), and repeat step C7.

C6. **Is \( \hat{n} \) in \( C^0 \)?** If \( S^C(\hat{n}',S_j) \neq \text{NIL} \), go to step C10. (This is the case when \( \hat{n} \) is not a prefix identifier of \( S(i) \).)

C7. **Find \( \hat{n} \)** If \( S^C(\hat{n}',S_j) \neq \text{NIL} \), set \( \hat{n} = S^C(\hat{n}',S_j) \) and go to step C8; otherwise, set \( S^C(\hat{n}',S_j) = * \), \( \hat{n} = F^2(\hat{n}') \), and repeat step C7.

C8. **Add \( \hat{n} \)** Add node \( \hat{n} \) by setting \( S^C(\hat{n}',S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n}) = F^2(\hat{n} ' = F^2(\hat{n}') \), and set \( S^C(\hat{n}',S_j) = \hat{n} \). Label \( \hat{n} \) by setting \( J_n = J_n + 1 \), \( L_g n + L_g (n+1) \), and \( L_g (k) = L_g (k) \) for \( 1 \leq k \leq t. \) (Note that \( L_g n + L_g (n+1) \).

C9. **Add \( \hat{n} \)** Add node \( \hat{n} \) by setting \( S^C(\hat{n}',S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n} ') = \hat{n} = F^2(\hat{n}') \), and \( S^C(\hat{n}',S_j) = \hat{n} \). Label \( \hat{n} \) by setting \( J_n = J_n + 1 \), \( L_g n + L_g (n+1) \), and \( L_g (k) = L_g (k) \) for \( 1 \leq k \leq t. \) Go to step C14.

C10. **[Insertion needed?]** If \( S^C(\hat{n}',S_j) \neq * \) and \( S^C(\hat{n}',S_j) \neq * \) and go to step C14; otherwise, set \( n+1 = \hat{n} ' \).

C11. **[Find father]** If \( S^C(n+1,S_j) \neq * \), set \( n+1 = S^C(n+1,S_j) \) and go to step C12. Else, set \( n+1 = F^2(n+1) \) and repeat step C11. (This step must terminate, by Lemma 3.)

C12. **[Insert \( \hat{n} \), if required]** If \( S^C(\hat{n}',S_j) \neq * \), go to step C13. Otherwise, set \( n+1 = S^C(n+1,S(\hat{n}+L_g(n))) \) and insert \( \hat{n} \) between \( n+1 \) and \( n+1 \) by setting \( S^C(n+1,S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n}) = n+1 \), \( S^C(\hat{n}',S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n}) = n+1 \), and \( S^C(\hat{n}',S_j) = \hat{n} \). Label \( \hat{n} \) by setting \( J_n = J_n + 1 \), \( L_g n + L_g (n+1) \), and \( L_g (k) = L_g (k) \) for \( 1 \leq k \leq t. \)

C13. **[Insert \( \hat{n} \), if required]** If \( S^C(\hat{n}',S_j) \neq * \), go to step C14. Otherwise, set \( n+1 = S^C(n+1,S(\hat{n}+L_g(n))) \) and insert \( n \) between \( n+1 \) and \( n+1 \) by setting \( S^C(n+1,S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n}) + n+1 \), \( S^C(\hat{n}',S(\hat{n}+L_g(n))) = \hat{n} = F^2(\hat{n}) + n+1 \), and \( S^C(\hat{n}',S_j) = \hat{n} \). Label \( \hat{n} \) by setting \( J_n = J_n + 1 \), \( L_g n + L_g (n+1) \), and \( L_g (k) = L_g (k) \) for \( 1 \leq k \leq t. \)

C14. **[Add \( n \)]** Add node \( n+1 \) by setting \( S^C(\hat{n}',S(\hat{n}+L_g(n))) = n+1 \), \( F^2(\hat{n}) + n+1 \), and \( S^C(\hat{n}',S_j) = n+1 \). Label \( n \) by setting \( J_n = J_n + 1 \), \( L_g n + L_g (n+1) \), and \( L_g (k) = L_g (k) \) for \( 1 \leq k \leq t. \) Stop.

Remark:

Several of the steps in Algorithm C could well be combined into a parametrized procedure.

To obtain the compacted prefix bi-tree \( C^0 \) for a string \( S \) of length \( m \), successively obtain \( C^0,C^0,\ldots,C^0 \), \( C^0 = C_p \). The run time of this procedure is of \( O(m) \), but in order to demonstrate this, we need to develop a few new ideas; step C11 of Algorithm C may be executed several times, and no labels are added, so the analysis used for Algorithm B does not apply.

Let \( C^0 = (\hat{n},n_0,n_1,n_2\ldots,n_{h+m-1}) \) be the compacted prefix bi-tree of \( S \) with \( n_0 \) having \( W(n_0) = 1(1) \). Define the Height of \( n \) \( h_0 \) to be the number of distinct ancestors of \( n \) in \( C^0 \). We wish to show that the number of steps executed in Algorithm C is in constructing \( C^0 \) is of \( O\left(h_0(d_0^0) \right) \), where \( d_0^0 = h_1(n_0+1) - h_1(n_0+1) \). (The constant insures that \( \delta_1 \geq 0. \) ) First, observe that Theorem 3 insures that every ancestor of \( n_0 \) except for \( n_0 \) has an s-father \( n_{h_1+1} \) which is an ancestor of \( n_{h_1} \). This implies that \( h_1(n_0) \leq h_1(n_{h_1+1})+1 \).

Next, observe that all steps in Algorithm C are executed exactly once, except possibly steps C5, C7, and C11. Steps C2 and C3 are executed \( h_1(n_{h_1+1})-h_1(n_{h_1+1}) \) times. In the case of a type 1 construction which requires insertion, step C11 is executed \( h_1(n_{h_1+1})-h_1(n_{h_1+1}) \) times; in the case of a type 2b or 2c construction, step C7 is executed \( h_1(n_{h_1+1})-h_1(n_{h_1+1}) \) times. Thus the total number of steps in Algorithm C is in constructing \( C^0 \) is of \( O\left(h_0(d_0^0) \right) \), where \( d_0^0 = h_1(n_0+1) - h_1(n_0+1) \). But \( h_0(n_i) \leq h_1(n_{h_1+1}+1) \), where \( n_i \) is the order \( h_1(n_{h_1+1}+1) \).

To prove that \( C^0 \) can be found in time \( O(m) \), observe \( \delta_1 = m+\delta_1(m) - \delta_1(n_0) \) since \( h_1(n_0) \leq m+1, \) \( \delta_1 = 0. \)
IV. Applications

In this section, we indicate how to use compacted prefix-trees to solve various pattern matching problems in linear time.

Problem 1: (Basic Pattern Matching Problem)

Given a string $S$ of length $k_1$, and pattern $P$ of length $k_2$, find all positions $i$ of $S$ such that $P = S(i : i + k_2 - 1)$.

Solution:

Append an end marker to $S$ and construct $T_C^P$ using Algorithm C. Determine whether any prefix of $P$ is a prefix of some prefix identifier of $S$, or vice versa by walking from the root of $T_C^P$ following branches labelled with symbols from $P$. If, at some stage of the walk, a node $n$ is reached at level $LG_n$ and labelled with $J_n$, check the value of $S(n, P(i : i + LG_n))$. If this value is NIL and if $n$ is not a leaf, then $P$ does not match $S$ anywhere. If $n$ is a leaf, then $P$ can possibly match $S$ at only one position, namely $J_n$. To see whether the match is valid, check for identity of $P(i : i + LG_n + k)$ and $S(J_n, P(i : i + LG_n + k))$ for $k = 0, \ldots, k_2 - LG_n - 1$. (If $k + k_2 - 1 \geq k_1$, no match exists.) Next, consider the case $S(n, P(i : i + LG_n)) = n'$. If $LG_n' = LG + 1$, simply continue the walk. On the other hand, if $LG_n = LG + q$, and $q > 1$, then it is necessary to compare $P(i : i + LG_n + k)$ and $S(J_n, P(i : i + LG_n + k))$ for $k = 1, \ldots, q - 1$. Lack of equality for any $k$ indicates no match; equality for all $k$ allows the walk to be continued. Finally, consider the case where a $LG_n = k_2$. In this event, each leaf in the sub-tree rooted at $n$ is labelled with the position of a match within $S$. A simple tree walk of this sub-tree finds these positions.

Problem 2: (Pattern match of several patterns with one string)

Given a string $S$ of length $k'$ and patterns $P_1, P_2, \ldots, P_q$ of lengths $k_1', k_2', \ldots, k_q'$, find all matches of each pattern in $S$.

Solution:

Simply walk each pattern individually through $T_C^P$ as in the solution to Problem 1. Note that the total effort is of $O(k' + k_1' + \ldots + k_q')$. Note also that the Knuth-Pratt-Morris technique does not fare as well for this problem, since every symbol of $S$ is examined $k'$ times. However, Karp [8] has extended their technique and has developed an alternate linear time solution to Problem 2.

Problem 3: (Internal Matching)

Given a string $S$ of length $k$, find for each position $i$ in $S$ another position $P(i)$ in $S$ such that the longest common prefix of $S_i$ and $S_{P(i)}$ of length $M(i)$ is no shorter than the longest common prefix of $S_i$ and $S_{j}$, $j \neq i$ and $j \neq P(i)$.

Solution:

Append an endmarker to $S$ and construct $T_C^P$. Locate (in constant time) the leaf $n$ labeled $J_n = i$. If $n'$ is a brother of $n$, then $P(i) = J_{n'}$, and the maximal match is of length $M(i) = LG_{n'} - 1$.

Problem 4: (External Matching)

Given two strings $S$ and $T$, of lengths $k$ and $k'$, find for every position $i$ in $S$ the position $P(i)$ and length $M(i)$ of the longest match $S(i : i + M(i) - 1) = T(P(i) : P(i) + M(i) - 1)$.

Solution:

Form the string $S = S_i S_{i+1} \ldots S_k$ where $i$: is a distinct separator symbol, and construct the compacted prefix-tree $T_C^P$ of $S$. We proceed as in the solution to Problem 3, except some care must be taken not to find matches entirely within $S$. Assume that $M(j)$ and $P(j)$ have been obtained for $i+1 \leq j \leq k$. To obtain $M(i)$ and $P(i)$, walk from the leaf $n$ labelled with $J_n = i$ towards the root until a node $n'$ is found with $J_{n'} > i$. (From the definition of $J$, $J_{n'} \geq J_n$, when $n'$ is an ancestor of $n$.) If $J_{n'} > T$, then $P(i) = J_{n'}$, and $M(i) = k'$; otherwise, $P(i) = P(J_{n'})$ and $M(i) = M(J_{n'})$. Note that the total number of steps in finding $P(i)$ and $M(i)$ for $1 \leq i \leq k'$ is of $O(k' + k)$ since each node in $T_C^P$ is examined at most two times. This construction of the match function $M$ and the position function $P$ has direct application to the File Transmission Problem, as discussed in [9]. Note also that Problems 1 and 2 can be solved with variants of this solution.

We leave it to the reader to work out the variants of our methods required to solve Problems 1 and 2 of [4] for strings.

Discussion:

The techniques of this paper do not appear powerful enough to solve directly some interesting related pattern matching problems. For example, when "don't-care" elements are introduced, the best known results [5] suggest that an $n \log n$ algorithm may be possible, but none has yet been found. Also, the "sub-sequence" problem, mentioned in [6], has, at present, only an $n^2$ solution.

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References

2. Knuth, Donald E., and Vaughan R. Pratt, "Automata theory can be useful," unpublished manuscript.
5. Paterson, M. S., "String-matching and other products," presented at a congress sponsored by the Istituto per le Applicazioni del Calcolo del


Figure 1. A bi-tree with 9 nodes.

Figure 2. Prefix-tree of S.
Type 1 construction.
(a)

Type 2a construction.
(b)

Type 2b construction.
(c)

Type 2c construction.
(d)

Figure 3.

$$T_4$$

$$T_3$$

$$S = S<1> \ldots S<7>$$

$$= 100100$$

Figure 4.
All $S_i$ branches shown (dotted lines) are labelled with $S<i>$.  

Figure 5a. Some branches from $B_{i+1}$.  

Figure 5b. Some branches from $B_i$.  

$S = S(1) \ldots S(17)$  
$= 0^4 1^4 0^4 1^4 1$  

Figure 6. A prefix-tree with $O(n^2)$ nodes.